EX-A

THEORETICAL ACOUSTICS

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4.3 SIMPLE-HARMONIC OSCILLATIONS

It has been seen in the last section that imposing boundary conditions limits the sorts of motion that a string can have, and that if the boundary conditions correspond to the fixing of both ends of the string to rigid supports, the motion is limited to *periodic* motion. The latter result is an unusual one, for we found in the last chapter that even as simple a system as a pair of coupled oscillators does not, in general, move with periodic motion. It is not unusual for a system to oscillate with simple-harmonic motion (which is a special type of periodic motion) when it is started off properly (we shall see that practically every vibrating system can do this); what is unusual in the string between rigid supports is that *every* motion is periodic, no matter how it is started.

Our problem in this section is to find the possible simple-harmonic oscillations of the string (the normal modes of vibration) and to see what the relation is between the frequencies of these vibrations that makes the resulting combined motion always periodic. The problem of determining the normal modes of vibration of a system is not just an academic exercise. For systems more complicated than that of the string between rigid supports, we have no method of graphical analysis similar to that of the last section, and the only feasible method of discussing the motion is to "take it apart" into its constituent simple-harmonic components. There is also a physiological reason for studying the problem, for the ear itself analyzes a sound into its simple-harmonic parts (if there are any). We distinguish between a note from a violin and a note from a bell, for instance, because of this analysis. If the frequencies present in a sound are all integral multiples of a fundamental frequency, as they are in a violin, the sound seems more musical than when the frequencies are not so simply related, as in the note from a bell.

Traveling and standing waves

We start our discussion with the wave equation (4.1.2), which, we showed, determines the motion of a perfectly flexible string as long as it is not displaced too far from equilibrium (as long as $|\partial y/\partial x| \ll 1$).

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad c^2 = \frac{T}{\epsilon} \tag{4.3.1}$$

The wave equation corresponds to a number of statements concerning the motion of a string. We saw in the last section that it implies that the wave motion travels with its shape unchanged, at a velocity c, independent of this shape. Since the derivative $\partial^2 y/\partial x^2$ is proportional to the curvature of the shape of the string at a given instant, Eq. (4.3.1) states that the *acceleration*

of any portion of the string is *directly proportional to the curvature* of that portion. If the curvature is downward, the acceleration is downward, and vice versa; and the greater the curvature, the faster the velocity changes.

If the string is infinite in extent, it can carry waves which travel exclusively in one direction. In that case, as was pointed out at the beginning of this chapter, if the time dependence of the wave is to be sinusoidal, its space dependence must also be sinusoidal. All simple-harmonic waves traveling in the positive x direction must have the form

$$y(x,t) = A\cos\left[\frac{\omega}{c}(x - ct) - \Phi\right]$$

or

$$y(x,t) = C\exp\left[\frac{i\omega}{c}(x - ct)\right] \tag{4.3.2}$$

if $C = Ae^{-i\Phi}$ and if physical meaning is attached only to the real part of the second expression. For a simple-harmonic wave in the negative x direction we substitute $-(x + ct)$ for $(x - ct)$ in these expressions. Incidentally, the reason we have chosen the time factor to be $e^{-i\omega t}$ rather than $e^{i\omega t}$ is that then the sign of the x part of the exponent, $e^{\pm i\omega x/c}$, indicates the direction of the wave.

For the wave of Eq. (4.3.2), the energy and momentum densities are [Eqs. (4.1.9) and (4.1.12)]

$$\text{Kinetic energy density} = U = \tfrac{1}{2}\epsilon\omega^2A^2\sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right]$$

$$\text{Potential energy density} = V = \tfrac{1}{2}T\left(\frac{\omega}{c}\right)^2A^2\sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right]$$

$$\text{Total energy density} = W_{tt} = \epsilon\omega^2A^2\sin^2\left[\frac{\omega}{c}(x - ct) - \Phi\right] = H \tag{4.3.3}$$

$$\text{Energy flux} = W_{tx} = cH$$

$$\text{Longitudinal momentum density} = W_{xt} = \frac{H}{c}$$

$$\text{Longitudinal stress} = W_{xx} = \frac{H}{c^2}$$

The energy density is greatest where the string's slope and transverse velocity are greatest, each packet of energy spaced a half wavelength from its neighbor, each traveling with a velocity c. Consequently, the energy flux W_{tx} is equal to c times the energy density W_{tt}. The wavelength of these waves is, of

course, the distance between one wave peak and the next, a distance such that an increase of x by λ will increase $(\omega/c)(x - ct)$ by 2π, so that $(\omega/c)\lambda = 2\pi$, or $\lambda = 2\pi c/\omega$.

We could reach the same conclusions by asking what sort of shape the string will have when it vibrates with simple-harmonic motion, i.e., when its time dependence is through the factor $e^{-i\omega t}$. Setting $y(x,t) = Y(x)e^{-i\omega t}$ into the equation of motion (4.1.3) or (4.1.8), we obtain a familiar equation for $Y(x)$.

$$\frac{d^2Y}{dx^2} + \left(\frac{\omega}{c}\right)^2 Y = 0 \qquad c^2 = \frac{T}{\epsilon} \tag{4.3.4}$$

which is identical with Eq. (1.2.1). This is the equation for simple-harmonic dependence on x, with "angular frequency" $k = \omega/c$ and "period" $\lambda = 2\pi/k = 2\pi c/\omega$. The quantity k is called the *wavenumber* of the wave; its dimensions are inverse length. The quantity λ is the wavelength of the wave, the distance from crest to crest of a sinusoidal wave traveling in one direction.

The general solution of Eq. (4.3.4) can be written

$$Y(x) = C_+e^{i\omega x/c} + C_-e^{-i\omega x/c}$$

so that

$$\begin{aligned} y(x,t) &= C_+e^{i\omega(x-ct)/c} + C_-e^{-i\omega(x+ct)/c} \\ &= A_+\cos\left[\frac{\omega}{c}(x - ct) - \Phi_+\right] + A_-\cos\left[\frac{\omega}{c}(x + ct) + \Phi_-\right] \end{aligned} \tag{4.3.5}$$

representing two waves, of the same frequency and wavelength, traveling in opposite directions along the string. Since the wave equation is linear, neither wave has any effect on the other.

This mutual independence of the waves extends to expressions for their energy-momentum-stress terms, such as the total energy,

$$H = U + V = \omega^2(\theta_+^2 + 2\theta_+\theta_- + \theta_-^2)$$

where

$$\theta_+ = A_+\sin\left[\frac{\omega}{c}(x - ct) - \Phi_+\right] \qquad \theta_- = A_-\sin\left[\frac{\omega}{c}(x + ct) + \Phi_-\right]$$

The mean value of the square terms is $\langle\theta_+^2\rangle = \frac{1}{2}A_+^2$ and $\langle\theta_-^2\rangle = \frac{1}{2}A_-^2$, neither of which is zero. But the cross terms can be written

$$\theta_+\theta_- = \tfrac{1}{2}A_+A_-\left[\cos(2\omega t + \Phi_+ + \Phi_-) - \cos\left(2\frac{\omega}{c}x - \Phi_+ + \Phi_-\right)\right]$$

When averaged over space and time, the average is zero. The energy flux has no cross term;

$$W_{tx} = -T\frac{\partial y}{\partial t}\frac{\partial y}{\partial x} = \epsilon c\omega^2(\theta_+^2 - \theta_-^2)$$

so that even the instantaneous values of the flux are simply the differences between the two individual fluxes. Thus the average values of the stress-energy tensor are

$$H = \langle W_{tt} \rangle = \tfrac{1}{2}\epsilon\omega^2(A_+^2 + A_-^2) = \frac{1}{c^2}\langle W_{xx} \rangle$$
$$\Upsilon = \langle W_{tx} \rangle = \tfrac{1}{2}\epsilon\omega^2 c(A_+^2 - A_-^2) = \frac{1}{c}\langle W_{xt} \rangle \qquad (4.3.6)$$

The energy and stress terms are the sum of the terms arising from each wave. The energy and momentum fluxes are the difference of the terms, since the two waves are flowing in opposite directions.

If the amplitudes of the two simple-harmonic waves are equal, there is no net flow of energy or momentum, and the combination is called a *standing wave*.

$$y(x,t) = C_+e^{i\omega(x-ct)/c} + C_-e^{-i\omega(x+ct)/c}$$
$$= 2A\cos\left(\frac{\omega}{c}x + \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right)\exp\left(-i\omega t + \tfrac{1}{2}\Phi_+ + \tfrac{1}{2}\Phi_-\right) \qquad \text{(real part)}$$
$$= 2A\cos\left(\frac{\omega}{c}x + \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right)\cos\left(\omega t - \tfrac{1}{2}\Phi_+ - \tfrac{1}{2}\Phi_-\right) \qquad (4.3.7)$$

where $C_+ = Ae^{i\Phi_+}$ and $C_- = Ae^{i\Phi_-}$ have the same amplitude but different phases. In this case the shape of the wave does not move along the string; it simply oscillates in amplitude with simple-harmonic motion. At points where $\cos[(\omega/c)x + \frac{1}{2}\Phi_+ - \frac{1}{2}\Phi_-] = 0$, the two traveling waves always cancel each other and the string never moves. These points are called the *nodal points* of the wave motion. In the case that we are considering, where the density and tension are uniform, the nodal points are equally spaced along the string a distance $c/2\nu$ apart, two for each wavelength. Halfway between each pair of nodal points is the part of the string having the largest amplitude of motion, where the two traveling waves always add their effects. This portion of the wave is called a *loop*, or *antinode*.

We should ask how a standing wave gets established and is maintained, for if there is no motion at each node, there can be no flow of energy from one loop to its neighbors. The answer is that a standing wave is a steady-state situation. During the transient state, when energy is being distributed along the string, the nodes are not perfect (that is, y is not exactly zero there) and energy does pass from one loop to the next. Also, even for the steady-state situation, the nodes are only perfect when there is no friction. With zero friction, once a loop has acquired its energy, it can oscillate forever. If friction is present, the "nodes" are simply places of minimal (but not zero) amplitude of vibration; some energy flows from loop to loop.

Normal modes

So far, we have neglected boundary conditions. If we require that $y = 0$ when $x = 0$, the general form of (4.3.5) can no longer be used; the number of possible harmonic motions is limited. The expression for y that must be used is the standing-wave form (4.3.7) with the angles Φ so chosen that *a nodal point coincides with the point of support* $x = 0$:

$$y = A \sin\left(\frac{2\pi\nu}{c} x\right) \cos(2\pi\nu t - \Phi) \tag{4.3.8}$$

This agrees with the discussion in the previous section. For the simple boundary condition that we have used, the reflected wave has the same amplitude as the incident wave; and when the incident one is sinusoidal, the result is a set of standing waves. Any frequency is allowed, however.

When the second boundary condition $y = 0$ at $x = l$ is added, the number of possible simple-harmonic motions is still more severely limited. For now, of all the possible standing waves indicated in (4.3.8), *only those which have a nodal point at* $x = l$ *can be used*. Since the distance between nodal points depends on the frequency, the string fixed at both ends cannot vibrate with simple-harmonic motion of any frequency; only a discrete set of frequencies is allowed, the set that makes $\sin[(2\pi\nu/c)l]$ zero. The distance between nodal points must be l, or it must be $l/2$ or $l/3$, etc. The allowed frequencies are therefore $c/2l$, $2c/2l$, $3c/2l$, etc., and the different allowed simple-harmonic motions are all given by the expression

$$y = A_n \sin\left(\frac{\pi n x}{l}\right) \cos\left(\frac{\pi n c}{l} t - \Phi_n\right) \qquad n = 1, 2, 3, 4, \ldots$$
$$\nu_n = \frac{nc}{2l} = \frac{n}{2l}\sqrt{\frac{T}{\epsilon}} \tag{4.3.9}$$

The lowest allowed frequency $\nu_1 = c/2l$ is called the *fundamental frequency* of vibration of the string. It is the frequency of the general periodic motion of the string, as we showed in the last section. The higher frequencies are called *overtones*, the first overtone being ν_2, the second ν_3, and so on.

The equation for the allowed frequencies given in Eq. (4.3.9) expresses an extremely important property of the uniform flexible string stretched between rigid supports. It states that the frequencies of all the overtones of such a string are *integral multiples of the fundamental frequency*. Overtones bearing this simple relation to the fundamental are called *harmonics*, the fundamental frequency being called the first harmonic, the first overtone (twice the fundamental) being the second harmonic, and so on.

Very few vibrating systems have harmonic overtones, but these few are the bases of nearly all musical instruments. For when the overtones are harmonic, the sound seems particularly satisfying, or musical, to the ear.

The general solution of this is

$$y = C_1e^{2\pi\mu x} + C_2e^{-2\pi\mu x} + C_3e^{2\pi i\mu x} + C_4e^{-2\pi i\mu x}$$
$$= a\cosh(2\pi\mu x) + b\sinh(2\pi\mu x) + c\cos(2\pi\mu x) + d\sin(2\pi\mu x) \quad (5.1.11)$$

where $\cosh u = \cos(iu)$ and $\sinh u = -i\sin(iu)$. See Eq. (1.2.10) and Tables I and II.

This general solution satisfies Eq. (5.1.10) for any value of the frequency ν. It is, of course, the boundary conditions that pick out the set of allowed frequencies.

Bar clamped at one end

For example, if we have a bar of length l clamped at one end $x = 0$, the boundary conditions at this end are that *both* y and its slope $\partial y/\partial x$ must be zero at $x = 0$. The particular combination of the general solution (5.1.11) that satisfies these two conditions is the one with $c = -a$ and $d = -b$.

$$Y = a[\cosh(2\pi\mu x) - \cos(2\pi\mu x)] + b[\sinh(2\pi\mu x) - \sin(2\pi\mu x)] \quad (5.1.12)$$

If the other end is free, y and its slope will not be zero, but the bending moment $M = QS\kappa^2(d^2Y/dx^2)$ and the shearing force $F = -QS\kappa^2(d^3Y/dx^3)$ must both be zero, since there is no bar beyond $x = l$ to cause a moment or a shearing stress. We see that *two* conditions must be specified for each end instead of just one, as in the string. This is due to the fact that the equation for Y is a fourth-order differential equation, and its solution involves four arbitrary constants whose relations must be fixed, instead of two for the string. It corresponds to the physical fact that whereas the only internal stress in the string is tension, the bar has two, bending moment and shearing force, each depending in a different way on the deformation of the bar.

The two boundary conditions at $x = l$ can be rewritten as $\dfrac{1}{4\pi^2\mu^2}\dfrac{d^2Y}{dx^2} = 0$ and $\dfrac{1}{8\pi^3\mu^3}\dfrac{d^3Y}{dx^3} = 0$ at $x = l$. Substituting expression (5.1.12) in these, we obtain two equations that fix the relationship between a and b and between μ and l:

$$a[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] + b[\sinh(2\pi\mu l) + \sin(2\pi\mu l)] = 0$$

$$a[\sinh(2\pi\mu l) - \sin(2\pi\mu l)] + b[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] = 0$$

or

$$b = a\frac{\sin(2\pi\mu l) - \sinh(2\pi\mu l)}{\cos(2\pi\mu l) + \cosh(2\pi\mu l)} = -a\frac{\cos(2\pi\mu l) + \cosh(2\pi\mu l)}{\sin(2\pi\mu l) + \sinh(2\pi\mu l)} \quad (5.1.13)$$

By dividing out a and multiplying across, we obtain an equation for μ:

$$[\cosh(2\pi\mu l) + \cos(2\pi\mu l)]^2 = \sinh^2(2\pi\mu l) - \sin^2(2\pi\mu l)$$

Utilizing some trigonometric relationships, this last equation can be reduced to two simpler forms:

$$\cosh(2\pi\mu l)\cos(2\pi\mu l) = -1 \qquad \text{or} \qquad \coth^2(\pi\mu l) = \tan^2(\pi\mu l) \tag{5.1.14}$$

where $\coth z = \cosh z/\sinh z$.

The allowed frequencies

We shall label the solutions of this equation in order of increasing value. They are $2\pi\mu_1 l = 1.8751$, $2\pi\mu_2 l = 4.6941$, $2\pi\mu_3 l = 7.8548$, etc. To simplify the notation, we let $1/\pi$ times the numbers given above have the labels β_n, so that

$$\mu_n = \frac{\beta_n}{2l} \tag{5.1.15}$$

where $\beta_1 = 0.597$, $\beta_2 = 1.494$, $\beta_3 = 2.500$, etc. It turns out that β_n is practically equal to $n - \frac{1}{2}$ when n is larger than 2.

By fixing μ, we fix the allowed values of the frequency. Using Eq. (5.1.10), we have

$$\nu_n = \frac{\gamma^2\mu_n^2}{2\pi} = \frac{\pi}{2l^2}\sqrt{\frac{Q\kappa^2}{\rho}}\,\beta_n^2 \tag{5.1.16}$$

or

$$\nu_1 = \frac{0.55966}{l^2}\sqrt{\frac{Q\kappa^2}{\rho}} \qquad \begin{aligned} \nu_2 &= 6.267\nu_1 \\ \nu_3 &= 17.548\nu_1 \\ \nu_4 &= 34.387\nu_1 \\ &\cdots\cdots\cdots \end{aligned}$$

Notice that the allowed frequencies depend on the inverse *square* of the length of the bar, whereas the allowed frequencies of the string depend on the inverse first power.

Equation (5.1.16) shows how far from harmonics are the overtones for a vibrating bar. The first overtone has a higher frequency than the sixth harmonic of a string of equal fundamental. If the bar were struck so that its motion contained a number of overtones with appreciable amplitude, it would give out a shrill and nonmusical sound. But since these high-frequency overtones are damped out rapidly, the harsh initial sound will quickly change to a pure tone, almost entirely due to the fundamental. A tuning fork can be considered to be two vibrating bars, both clamped at their lower ends. The fork exhibits the preceding behavior, the initial metallic "ping" rapidly dying out and leaving an almost pure tone.

The characteristic functions

The characteristic function corresponding to the allowed frequency ν_n is given by the equation

$$\psi_n = a_n\left(\cosh\frac{\pi\beta_n x}{l} - \cos\frac{\pi\beta_n x}{l}\right) + b_n\left(\sinh\frac{\pi\beta_n x}{l} - \sin\frac{\pi\beta_n x}{l}\right) \tag{5.1.17}$$

where

$$-b_n = a_n \frac{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}{\sinh(\pi\beta_n) + \sin(\pi\beta_n)} = a_n \frac{\sinh(\pi\beta_n) - \sin(\pi\beta_n)}{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}$$

We shall choose the value of a_n so that $\int_0^l \psi_n^2\,dx = l/2$, by analogy with the sine functions for the string. The resulting values for a_n and b_n are $a_1 = 0.707$, $b_1 = -0.518$, $a_2 = 0.707$, $b_2 = -0.721$, $a_3 = 0.707$, $b_3 = -0.707$, etc. For n larger than 2, both a_n and b_n are practically equal to $1/\sqrt{2}$. Some of the properties of these functions that will be of use are

$$\int_0^l \psi_m(x)\psi_n(x)\,dx = \begin{cases} 0 & m \neq n \\ \dfrac{l}{2} & m = n \end{cases} \qquad \psi_n(l) = (-1)^{n-1}\sqrt{2}$$

$$\left(\frac{d\psi_1}{dx}\right)_{x=l} = 1.040\,\frac{\pi\beta_1}{l} \qquad \left(\frac{d\psi_2}{dx}\right)_{x=l} = -1.440\,\frac{\pi\beta_2}{l} \tag{5.1.18}$$

$$\left(\frac{d\psi_n}{dx}\right)_{x=l} \simeq (-1)^{n-1}\sqrt{2}\,\frac{\pi\beta_n}{l} \qquad \text{and} \qquad \beta_n \simeq n - \tfrac{1}{2} \qquad n > 2$$

$$\psi_n \simeq \frac{1}{\sqrt{2}}\left[e^{-\pi\beta_n x/l} + (-1)^{n-1}e^{\pi\beta_n(x-l)/l}\right] + \sin\left(\frac{\pi\beta_n x}{l} - \frac{\pi}{4}\right) \qquad n > 2$$

The shapes of the first five characteristic functions are shown in Fig. 5.4. Note that for the higher overtones most of the length of the bar has the sinusoidal shape of the corresponding normal mode of the string, with the nodes displaced toward the free end. In terms of the approximate form given above for ψ_n, the sine function is symmetrical about the center of the bar; the first exponential alters the sinusoidal shape near $x = 0$ enough to make ψ_n have zero value and slope at this point; and the second exponential adds enough near $x = l$ to make the second and third derivatives vanish. Note also that the number of nodal points in ψ_n is equal to $n - 1$, as it is for the string.

In accordance with the earlier discussion of series of characteristic functions, we can now show that a bar started with the initial conditions, at $t = 0$, of $y = y_0(x)$ and $\partial y/\partial t = v_0(x)$ will have a subsequent shape given by the series

$$y = \sum_{n=1}^{\infty} \psi_n(x)[B_n \cos(2\pi\nu_n t) + C_n \sin(2\pi\nu_n t)] \tag{5.1.19}$$

where

$$B_n = \frac{2}{l}\int_0^l y_0(x)\psi_n(x)\,dx$$

$$C_n = \frac{1}{\pi\nu_n l}\int_0^l v_0(x)\psi_n(x)\,dx$$